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The Beliaev–Zelivinsky boson expansion and particle–vibration coupling methods in an exactly soluble model

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Abstract. The Beliaev–Zelivinsky boson expansion and particle–vibration coupling methods are studied in comparison with an exact treatment of the monopole model. Two of the three boson expansions considered were valid for all interaction strengths. A third expansion and the particle–vibration coupling method were found to be equivalent to perturbation theory. The difficulty associated with the overcompleteness of the basis in the particle–vibration coupling method was resolved both in the computation of the energies and the matrix element of the monopole specific operator.

1. Introduction

The observation that the collective excitations of many-body systems often give rise to a harmonic spectrum has led to the development of theories which incorporate the use of phonon coordinate variables from the outset. One of the simplest examples is the random phase approximation. However, since the random phase approximation diverges as the phase transition is approached, various alternative theories, or improvements to the random phase approximation, have been proposed. A widely used model against which such approximations have been tested is that introduced by Lipkin *et al* (1965). We work with a slight generalization of the Lipkin model called the monopole model, also discussed in the previous reference, because the former is trivial in the (fundamental) Tamm–Dancoff approximation which is of some interest to us here.

It is our purpose to investigate the Beliaev–Zelivinsky boson expansion (Beliaev and Zelivinsky 1962) and particle–vibration coupling methods with this model. These methods will be abbreviated by BZ and PVC respectively. A similar study was done with the Lipkin model for both the BZ and Marumori boson expansion methods (Marumori *et al* 1964)¶. We note, however, that this treatment did not exactly follow the BZ method as discussed in § 3 because the Hamiltonian was normal ordered in the boson operators. The solutions were then found to diverge as the phase transition was approached.

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¶ The relation between the two methods is discussed quite clearly by Marshalek (1971), Janssen *et al* (1971) and Li *et al* (1971).

Since this result may cast doubt on the utility of the BZ method, and to consider other points of interest associated with this theory, we feel it is important to note that an adherence to the BZ method gives very good results for all values of the interaction strength in the monopole model.

The monopole model is briefly outlined in § 2. In § 3, three types of BZ expansion are considered; the Holstein–Primakoff (HPE), the Tamm–Dancoff (TDE) and the random phase (RPE). In § 4 the PVC method is considered, while all numerical results are presented in § 6. A comparison of the various methods is made in § 5.

2. The monopole model

The monopole model Hamiltonian on two levels is defined as

$$H = \frac{\epsilon}{2}\sum_{m,\sigma}\sigma a^{\dagger}_{m,\sigma}a_{m,\sigma} - \frac{V}{2}\sum_{\substack{m,m'\\ \sigma,\sigma'}} a^{\dagger}_{m,\sigma}a^{\dagger}_{m',\sigma'}a_{m',-\sigma'}a_{m,-\sigma} \tag{1}$$

where $a^{\dagger}_{m,\sigma}$ creates a fermion with quantum numbers $(m, \pm 1)$ in the $\binom{\text{upper}}{\text{lower}}$ level, m serving to denote the degenerate states of which there are $2\Omega = 2j+1$ within each level. Its general feature is that pairs of particles are scattered between the levels without changing their values of m. Thus, with the definition that the parity π of a state is (even, odd) if the population of the lower level ($\sigma = -1$) is (even, odd), it is observed that the states of different parities are not connected and the energy matrix splits into two. Defining the four physical operators

$$A^{\dagger} = \frac{1}{\sqrt{(2\Omega)}}\sum_m a^{\dagger}_{m,1}a_{m,-1}, \qquad A = \frac{1}{\sqrt{(2\Omega)}}\sum_m a^{\dagger}_{m,-1}a_{m,1} \tag{2a}$$

and

$$B_1 = \frac{1}{\sqrt{\Omega}}\sum_m a^{\dagger}_{m,1}a_{m,1}, \qquad B_{-1} = \frac{1}{\sqrt{\Omega}}\sum_m a_{m,-1}a^{\dagger}_{m,-1} \tag{2b}$$

the monopole Hamiltonian may be re-expressed as

$$H = \epsilon\left(\frac{\sqrt{\Omega}}{2}(B_1+B_{-1})-\Omega\right) - V\Omega(A^{\dagger}+A)^2 + V\Omega \tag{3}$$

where we have chosen, as always unless otherwise stated, the number of particles N to be equal to 2Ω, that is, in the uncorrelated ground state ($V = 0$) the lower level is filled and the upper level empty. The physical operators obey the commutation relations

$$[A, A^{\dagger}] = 1 - \frac{B_1+B_{-1}}{2\sqrt{\Omega}} \tag{4a}$$

$$[A, B_{\pm 1}] = \frac{A}{\sqrt{\Omega}}. \tag{4b}$$

An exact solution of the Hamiltonian makes use of the correspondence of the physical operators to the operators

$$T_0 = \frac{\sqrt{\Omega}}{2}(B_1+B_{-1})-\Omega, \qquad T_+ = \sqrt{(2\Omega)}A^{\dagger}, \qquad T_- = T_+^{\dagger} \tag{5}$$

which, from the relations (4), obey the commutation relations

$$[T_+, T_-] = 2T_0, \qquad [T_0, T_\pm] = \pm T_\pm . \tag{6}$$

The T operators are then identified as the generators of R_3 and one may proceed to obtain the properties of H as was done by Lipkin *et al* (1965). For example, the states may be labelled by the total isospin T and its projection M_T. Using the discussion found therein, limiting values of various interesting quantities may be calculated. These prove to be useful when evaluating numerical results since they indicate the dominance of one interaction over the other. Thus in the monopole limit ($\epsilon = 0$), the ground and first excited state energies (both of which could be degenerate) are

$$\begin{aligned} E_0 &= -V\Omega(2\Omega-1) \\ E_1 &= -V\Omega\left(2\Omega-5+\frac{2}{\Omega}\right) \qquad (\text{for } \epsilon = 0,\ V \neq 0). \end{aligned} \tag{7}$$

A convenient quantity with which the state of the system can be pictured very readily is the occupation number of the upper level in the ground state divided by Ω, simply called the 'normalized' occupation number $\tilde{n}$, and which has the obvious limits (when $N = 2\Omega$)

$$\tilde{n} = \begin{cases} 1 & \text{for } \epsilon = 0,\ V \neq 0 \\ 0 & \text{for } \epsilon \neq 0,\ V = 0. \end{cases} \tag{8}$$

The absolute value squared of the matrix element of the monopole specific operator, $\sigma_M(\Omega)$, is defined as

$$\sigma_M(\Omega) = |\langle \text{first}\,; 2\Omega|(A^\dagger + A)|\text{ground}\,; 2\Omega\rangle|^2 \tag{9}$$

with the notation that $|\text{ground}; 2\Omega\rangle$ is the eigenvector of the ground state of the 2Ω particle system, etc. $\sigma_M(\Omega)$ has the limits

$$\sigma_M(\Omega) = \begin{cases} \Omega & \text{for } \epsilon = 0,\ V \neq 0 \\ 1 & \text{for } \epsilon \neq 0,\ V = 0. \end{cases} \tag{10}$$

The one body transfer operator $a^\dagger_{jm\sigma}$ has fractional parentage coefficients between the states of 2Ω and $2\Omega+1$ particles

$$\begin{aligned} &\langle T, M_T; j, m; 2\Omega+1|a^\dagger_{jm\sigma}|T', M'_T; j', m'; 2\Omega\rangle \\ &\quad = \langle \Omega-\tfrac{1}{2}, M_T; j, m; 2\Omega+1|a^\dagger_{jm\sigma}|\Omega, M_T; 0, 0; 2\Omega\rangle \\ &\quad = \pm\left(\frac{\Omega \mp M_T}{2\Omega}\right)^{1/2} \qquad \text{for } \sigma = \pm 1. \end{aligned} \tag{11}$$

Using equation (11), the one body transfer cross section $\sigma_1(S_1, S_2)$ from state 1 of the 2Ω particle system to state 2 of the $2\Omega+1$ particle system may be calculated.

$$\sigma_1(S_1, S_2) = |\langle S_2; 2\Omega+1|a^\dagger_{jm\sigma}|S_1; 2\Omega\rangle|^2 \begin{pmatrix} \delta_{\pi_1,\pi_2} \\ \delta_{\pi_1+1,\pi_2} \end{pmatrix}, \qquad \sigma = \pm 1 \tag{12}$$

where the δ functions indicate that $\sigma_1(S_1, S_2)$ will be zero if a particle is transferred into the upper (lower) level between states of different (same) parity. In particular, if S_1 and S_2

are the ground states of their respective systems then

$$\sigma_1(\mathrm{g}, \mathrm{g}) = \begin{cases} \frac{1}{2} & \text{for } \epsilon = 0, V \neq 0 \\ 0 & \text{for } \epsilon \neq 0, V = 0. \end{cases} \tag{13}$$

We will also calculate σ_1 when S_1 represents the ground state and S_2 the first excited state (f), or the second excited state (s).

3. Beliaev–Zelivinsky expansions

The appealing feature of the BZ method is that it yields an expansion in terms of a smallness parameter, here Ω^{-1}, and more generally the inverse of the number of states available to the active particles. As originally formulated the method may violate exclusion principle constraints and therefore it is not clear how this violation may effect the solutions. Marshalek (1971) has pointed out a means of satisfying these constraints within the BZ method. Likewise, it is known that the Marumori boson expansions as originally formulated do not readily converge, although they do take into account all exclusion principle requirements. Li *et al* (1971) have recently given a modified Marumori expansion which is developed in a smallness parameter. These problems are of no concern here since, as was shown by Pang *et al* (1968), both methods lead to the same expansion for the Lipkin (or monopole) model. The only effect of the exclusion principle is to limit the size of the basis. We may thus focus attention on the convergence properties of the BZ method.

Pairs of fermion operators may be expressed as an expansion in powers of the boson operators $\alpha, \alpha^{\dagger}$ where the boson operators† satisfy the commutation relations $[\alpha, \alpha^{\dagger}] = 1$ and $[\alpha, \alpha] = 0$. The resultant operator $(F)_{\mathrm{B}}$ is called the boson image of the original fermion operator F. In the method of Beliaev and Zelivinsky the boson images are required to satisfy the same commutation relations as the original fermion operators. It is clear from the relations (4) that the boson images of B_{-1} and B_1 can differ by at most a constant. In the present case, there being 2Ω particles, this constant is zero.

The boson images act in a many boson space, the functions of which are in one to one correspondence with the original many fermion functions. The identification of this physical boson space is not obvious in general. In the present model, however, it presents no difficulty.

We shall consider three different expansions.

(i) Holstein–Primakoff expansion (HPE)

$$(A^{\dagger})_{\mathrm{B}} = \alpha^{\dagger} \sum_{\mu=0}^{\infty} g_{\mu} \hat{n}^{\mu}$$

$$(B)_{\mathrm{B}} = \sum_{\mu=0}^{\infty} h_{\mu} \hat{n}^{\mu} \tag{14}$$

where $\hat{n} = \alpha^{\dagger}\alpha$ and $(B)_{\mathrm{B}}$ is the common boson image of $B_{\pm 1}$. The coefficients in equations (14) follow from an expansion in powers of $\hat{n}$ of the exact boson images of the T operators

† One type of boson is sufficient but not necessary to solve the monopole model. The latter has been solved by Evans and Kraus (1972) using four independent bosons.

given by Holstein and Primakoff (1940);

$$(T_0)_B = \hat{n} - T$$
$$(T_+)_B = (T_-^\dagger)_B = \alpha^\dagger (2T - \hat{n})^{1/2} \tag{15}$$

where T is an integer or semi-integer satisfying $2T \leqslant \min(N, 4\Omega - N)$. The boson number n, varies from zero to $2T$, so that the expansions (14) converge. An n boson function labelled by T, corresponds to an N fermion function with spin symmetry $[2^{\frac{1}{2}N - T}, 1^{2T}]$. In our case $N = 2T = 2\Omega$, so that the corresponding spin function is unique having total spin zero.

Using equations (5) and (15) the boson image of the Hamiltonian (3) may be written

$$(H)_B = (\hat{n} - \Omega)\epsilon + (\hat{n}^2 - 2\Omega\hat{n})V - V\Omega \left\{ \alpha^\dagger \left(1 - \frac{\hat{n}}{2\Omega} \right)^{1/2} \alpha^\dagger \left(1 - \frac{\hat{n}}{2\Omega} \right)^{1/2} + \mathrm{hc} \right\}^2 \tag{16}$$

and from this a series expansion in Ω^{-1} for the energy eigenvalues may be obtained. In particular the lowest excitation energy is given by

$$\frac{\Delta E}{\epsilon} = 1 - \left(2 - \frac{1}{\Omega}\right)\left(\frac{V\Omega}{\epsilon}\right) - \left(2 - \frac{4}{\Omega} + \frac{3}{2\Omega^2}\right)\left(\frac{V\Omega}{\epsilon}\right)^2 - \left(4 - \frac{18}{\Omega} + \frac{20}{\Omega^2}\right)\left(\frac{V\Omega}{\epsilon}\right)^3 + \mathrm{O}(\Omega^{-3}). \tag{17}$$

Thus a boson Hamiltonian is available independent of the method of Beliaev and Zelivinsky which, in addition to providing the exact solution, directly yields expansions in Ω^{-1}.

(ii) Tamm–Dancoff expansion (TDE)

$$(A^\dagger)_B = \alpha^\dagger \sum_{\mu=0}^{\infty} b_\mu (\alpha^\dagger)^\mu \alpha^\mu$$
$$(B)_B = \sum_{\mu=0}^{\infty} C_\mu (\alpha^\dagger)^\mu \alpha^\mu. \tag{18}$$

Unlike the HPE, the TDE is normal ordered. Both expansions have the common property that the boson vacuum corresponds to the fermion state with no particle–holes present. Since $(B)_B$ measures this last quantity it is apparent that $C_0 = 0$. Moreover, one obtains

$$C_\mu = \frac{\delta_{\mu,1}}{\Omega}$$
$$b_0 = 1$$
$$b_1 = -1 + \left(1 - \frac{1}{2\Omega}\right)^{1/2} = \mathrm{O}(\Omega^{-1})$$
$$b_2 = -\tfrac{1}{2}(1 + 2b_1) + \tfrac{1}{2}(1 + 4b_1 + 2b_1^2) = \mathrm{O}(\Omega^{-2})$$
$$\vdots$$
$$b_l = \mathrm{O}(\Omega^{-l}). \tag{19}$$

We note that there are just the necessary number of equations to determine all the coefficients. Using these coefficients, the boson image of the Hamiltonian reads

$$(H)_B = (\hat{n}-\Omega)\epsilon - V\Omega\{2\hat{n}+1+\alpha^2+(\alpha^\dagger)^2+(4b_1-b_1^2-6b_2)\hat{n}^2+b_1(\alpha^2+(\alpha^\dagger)^2)$$
$$+(2b_1+b_1^2)(\hat{n}\alpha^2+(\alpha^\dagger)^2\hat{n})+(b_1^2+2b_2)(\hat{n}+2\hat{n}^3+\hat{n}\alpha^2+(\alpha^\dagger)^2\hat{n})\}+O(\Omega^{-3}). \tag{20a}$$

Expanding the coefficients to Ω^{-2}

$$(H)_B \simeq (\hat{n}-\Omega)\epsilon + V\Omega\left(1-2\hat{n}+\frac{\hat{n}^2}{\Omega}\right)$$
$$-V\Omega\left\{\left(1-\frac{1}{4\Omega}-\frac{1}{32\Omega^2}\right)(\alpha^2+(\alpha^\dagger)^2)-\frac{1}{2\Omega}(\hat{n}\alpha^2+(\alpha^\dagger)^2\hat{n})\right\}. \tag{20b}$$

Using the Hamiltonian (20) and perturbation theory, an expansion of the energy in powers of Ω^{-1} is obtained. We thus may reproduce, for instance, expression (17) for the lowest excitation energy. Alternatively, we may diagonalize either (20*a*) using (19) or (20*b*) within the physical boson basis. Both of these diagonalizations effectively include some higher order terms in Ω^{-1}.

Although both (20*a*) and (20*b*) were truncated at b_2 they differ by the fact that the coefficients in (20*a*) involve an infinite series in powers of Ω^{-1}. These may cancel with higher powers in the terms which were cut off. In (20*b*), however, all contributions to Ω^{-2} were retained and all higher orders discarded.

The relation between the coefficients of HPE and TDE for the Lipkin model is given by Pang *et al* (1968). In this reference, the commutation relations are satisfied by a TDE as in equation (18) and then the Hamiltonian is normal ordered. Thus the coefficient of a term in $(H)_B$ containing v bosons is made up to contributions of all powers of Ω^{-1} (larger than $\frac{1}{2}v-1$). An alternative procedure which is more consistent with the original idea of Beliaev and Zelivinsky is to fix the maximum order of Ω^{-1} and use as many bosons as required with that power of Ω^{-1}. Thus if we admit up to the power $\Omega^{(1-\frac{1}{2}v)}$, terms up to v phonons are to be included in the Hamiltonian. It is our conclusion, based on numerical calculations, that the latter procedure gives consistently good results for the monopole force (see § 6) and, in addition, it is probably easier to generalize for more complicated situations.

The matrix element of the monopole specific operator may also be calculated using perturbation theory. To Ω^{-1} and second order in the interaction it is the sum of

$${}_B\langle\psi_1|(A^\dagger)_B|\psi_0\rangle_B = 1+\left(\frac{1}{2}-\frac{1}{\Omega}\right)\left(\frac{V\Omega}{\epsilon}\right)^2 \tag{21a}$$

and

$${}_B\langle\psi_1|(A)_B|\psi_0\rangle_B = \left(1-\frac{1}{2\Omega}\right)\left(\frac{V\Omega}{\epsilon}\right)+\left(2-\frac{3}{\Omega}\right)\left(\frac{V\Omega}{\epsilon}\right)^2 \tag{21b}$$

(iii) Random phase expansion (RPE).

Another method which has been widely used (Beliaev and Zelivinsky 1962, Pang *et al*

1968, Sorensen 1968, 1969, Bès and Dussel 1969) is to make a canonical transformation

$$\alpha^\dagger = \lambda\beta^\dagger + \mu\beta \tag{22}$$

where λ and μ are given by the random phase approximation.

A more natural generalization of the random phase approximation (RPA) is obtained by allowing more terms in the expansions:

$$(A^\dagger)_B = \sum_{\mu=1}^{\infty} \sum_{\sigma=0}^{\mu} b_{\mu-\sigma,\sigma}(\alpha^\dagger)^{\mu-\sigma}\alpha^\sigma$$

$$(B)_B = \sum_{\mu=1}^{\infty} \sum_{\sigma=0}^{\mu} c_{\mu-\sigma,\sigma}(\alpha^\dagger)^{\mu-\sigma}\alpha^\sigma. \tag{23}$$

The commutation relation (4b) determines the coefficients $c_{\mu-\sigma,\sigma}$ (except c_{00}) as functions of the $b_{\mu-\sigma,\sigma}$. In lowest order,

$$c_{20} = c_{02} = \frac{b_{10}b_{01}}{\sqrt{\Omega}}$$

$$c_{11} = \frac{b_{10}^2 + b_{01}^2}{\sqrt{\Omega}}. \tag{24}$$

The relation (4a) implies a constraint between b_{10} and b_{01}.

$$b_{10}^2 - b_{01}^2 = 1. \tag{25}$$

Use has been made of the fact that c_{00} is of lower order than unity, since the number of particle–hole pairs in the ground state must be a small number.

In lowest order the Hamiltonian $(H)_B$ has two terms (apart from a constant). The first is proportional to $\hat{n}$ and the second one to $\alpha^2 + \alpha^{\dagger 2}$. Since one of the bs in equation (25) is still undetermined, we use it in order to diagonalize $(H)_B$. This supplies a further constraint

$$\epsilon b_{01}b_{10} = V\Omega(b_{01}+b_{10})^2 \tag{26}$$

and, not surprisingly, the RPA result:

$$(H)_B = \omega\hat{n}, \qquad \omega \equiv \epsilon\left(1 - 4\frac{V\Omega}{\epsilon}\right)^{1/2}. \tag{27}$$

Moreover,

$$b_{10} = \frac{\epsilon+\omega}{2\sqrt{(\epsilon\omega)}}, \qquad b_{01} = \frac{\epsilon-\omega}{2\sqrt{(\epsilon\omega)}}. \tag{28}$$

In the next order a similar procedure leads to

$$(H)_B = \left[\omega + \frac{(\epsilon^2-\omega^2)^2}{16\Omega\epsilon\omega}\left\{1 + 2\left(\frac{\epsilon-\omega}{\epsilon+\omega}\right)\right\}\right]\hat{n} + \frac{\epsilon^2-\omega^2}{16\Omega\epsilon\omega}(3\epsilon^2+\omega^2)\hat{n}^2. \tag{29}$$

In order to obtain this, c_{00} has been evaluated from the RPA correlated ground state (Sanderson 1965). Expanding ω from its RPA value (27) yields the result in (17) to order Ω^{-1}, which was consistently retained in deriving (29).

It is easily seen that continuing in this manner with the RPE always produces a diagonal Hamiltonian and a consistent expansion in Ω^{-1}. It is also clear that a finite expansion in Ω^{-1} will not converge for $\omega \sim 0$.

4. Particle–vibration coupling

An alternative approach to the boson expansion techniques has been suggested by Mottelson (1968). The method involves the use of a phonon plus a single-particle Hamiltonian which are coupled through an interaction which is linear in the boson operator. Thus in zeroth order in the interaction, the particle and phonon degrees of freedom are independent of each other. The Hamiltonian is given by

$$
\begin{aligned}
H &= H_0 + H_c \\
H_0 &= \omega\alpha^\dagger\alpha + \frac{\epsilon}{2}\sum_m \sigma\alpha^\dagger_{m,\sigma}a_{m,\sigma} \\
H_c &= -\Lambda\sqrt{(2\Omega)}(\alpha^\dagger+\alpha)(A^\dagger + A).
\end{aligned}
\tag{30}
$$

Here, ω is the phonon energy and Λ is the particle–vibration coupling constant. These two parameters are functions of V and Ω, which will be determined through the RPA in order to compare with low order perturbation theory.

This method differs from older treatments by the introduction of the vertices 1(c) and 1(d) of figure 1. For the particular case of the monopole interaction, the usual vertices 1(a) and 1(b) vanish.

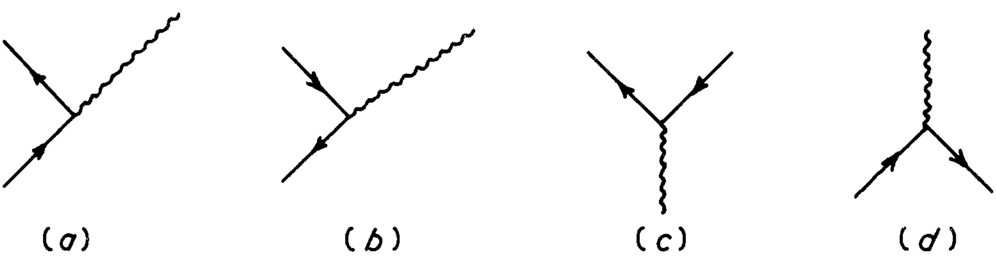

Figure 1. Particle–vibration interaction vertices.

In order to calculate ω we consider the linearized commutator of $A^\dagger$ with the Hamiltonian (3)

$$
[H, A^\dagger] \simeq \epsilon A^\dagger - (2\Omega-1)VA^\dagger - (2\Omega-1)VA. \tag{31}
$$

Solution of the coupled eigenvalue equations for $A^\dagger$, A then gives the values of equation (27) in the limit $\Omega \to \infty$ and $V\Omega$ finite. However, equation (31) contains important terms of order V as distinct from $V\Omega$. Of these, the diagonal one is just the Fock correction to the energy of a particle with $\sigma = +1$. The off-diagonal term in this order arises from the exchange interaction at vertices where two particle–hole pairs are simultaneously created or destroyed. These effects are due to the terms neglected when treating the monopole pair operator $A^\dagger$ as a true boson creator and are therefore absent in the boson approximation (Lane 1964). However, they represent physical effects contributing to the order in which we are working and would give rise to additional graphs if not included in the RPA. It is interesting to note that these contributions plus all others of the same order are automatically taken into account by the BZ method detailed in the previous section. Inclusion of these terms leads to

$$\omega = \epsilon\left\{1-4\frac{V\Omega}{\epsilon}\left(1-\frac{1}{2\Omega}\right)\right\}^{1/2}. \tag{32}$$

It is clear that if

$$\omega_0 = \epsilon\left(1-4\frac{V\Omega}{\epsilon}\right)^{1/2}$$

then

$$\begin{aligned}\omega &\simeq \omega_0+\frac{\epsilon V}{\omega_0}\\ &= \epsilon\left\{1-2\left(\frac{V\Omega}{\epsilon}\right)-2\left(\frac{V\Omega}{\epsilon}\right)^2-4\left(\frac{V\Omega}{\epsilon}\right)^3-\dots\right\}\\ &\quad+\frac{\epsilon}{\Omega}\left\{\frac{V\Omega}{\epsilon}+2\left(\frac{V\Omega}{\epsilon}\right)^2+6\left(\frac{V\Omega}{\epsilon}\right)^3+\dots\right\}.\end{aligned} \tag{33}$$

The second order contribution to the energy of the phonon corresponds to the graphs given in figures 2(*a*) and 2(*b*). However, the processes represented there have been taken into account in the definition of the phonon and therefore should not be

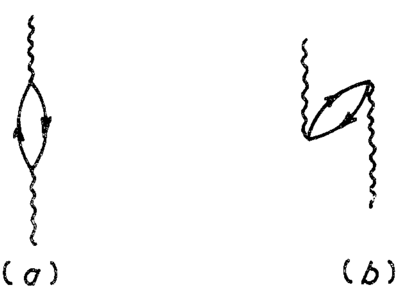

Figure 2. Graphs which do not contribute to the phonon energy.

included. Quite generally, we suppress all diagrams in which the phonon is decomposed into its components and composed again without any interaction affecting the particle–hole components. The lowest order contributing graphs occur in fourth order. In this order there are two types of graph. The first one (figure 3(*a*)) corresponds to the exchange between the particle (hole) in a vacuum fluctuation. The second type (figure 3(*b*)) represents the process through which (i) the particle in the component of a phonon falls into the hole of a component of a vacuum fluctuation by emitting a phonon and (ii) this phonon together with the remaining pair is mixed again with the vacuum.

To proceed with the calculation of the contribution of these graphs Λ must be determined. This can be done by requiring that the matrix element of the specific operator $A^\dagger + A$ between the phonon vacuum and the one phonon state should be given by the RPA value $(\epsilon/\omega)^{1/2}$†. From graphs 4(*a*) and 4(*b*) we conclude that

$$\frac{(8\Omega)^{1/2}\epsilon\Lambda}{\epsilon^2-\omega^2} = \left(\frac{\epsilon}{\omega}\right)^{1/2}$$

† To lowest order in Ω^{-1}.

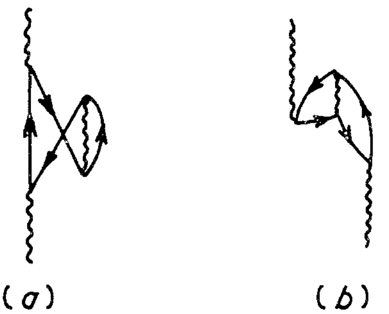

Figure 3. Fourth order contributions to the energy.

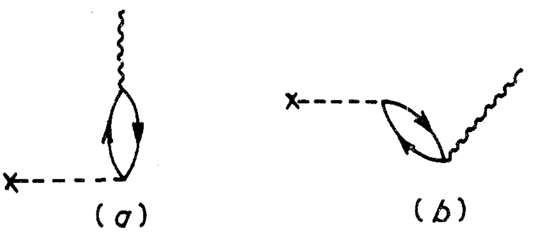

Figure 4. Lowest order graphs for the specific operator matrix element.

hence

$$\Lambda = \frac{\epsilon^2 - \omega^2}{(8\Omega\epsilon\omega)^{1/2}}. \tag{34}$$

As a check on consistency we may calculate the energy correction given by graphs 2(*a*) and 2(*b*) using the value of Λ in equation (37). The result is easily found to be $-(\epsilon^2 - \omega^2)/2\omega$ which differs from $-(\epsilon - \omega)$ by a term proportional to V^2. This implies consistency to order V. Summing the contributions of all the different time orderings implied by the graphs shown in figure 3(*a*) we find

$$\frac{8\Omega\Lambda^4\epsilon}{(\epsilon^2 - \omega^2)^2} = \frac{2\Omega\epsilon V^2}{\omega^2} = \frac{2\epsilon}{\Omega}\left(\frac{V\Omega}{\epsilon}\right)^2 + \frac{8\epsilon}{\Omega}\left(\frac{V\Omega}{\epsilon}\right)^3 + \dots$$

while those implied by figure 3(*b*) give

$$\frac{4\Omega\Lambda^4}{\epsilon(\epsilon^2 - \omega^2)} = \frac{4\Omega^2 V^3}{\omega^2} = \frac{4\epsilon}{\Omega}\left(\frac{V\Omega}{\epsilon}\right)^3 + \dots.$$

Adding these expressions to that already found for ω reproduces the result given by equation (17) to order $(V\Omega/\epsilon)^3$ and Ω^{-1}.

The two degrees of freedom bear a deeper relation to each other than that described by the interaction between them. Although they were treated in lowest order as being independent, in fact the phonon mode is itself built up from particle excitations so that there is an essential redundancy in the description of the states of the system. A prescription must therefore be found to treat the two degrees of freedom in a consistent manner. This is in contrast, for example, to the case of the electron–phonon system in a metallic lattice. For this reason we must use in place of $A^\dagger$ an effective monopole pair operator which acts on both degrees of freedom. By requiring that the matrix elements between the zero and one phonon states be as given by the RPA we deduce

$$(A^\dagger)_{\rm eff} = A^\dagger + x\alpha^\dagger + y\alpha \tag{35}$$

where x and y are the matrix elements represented by the graphs in figures 4(a) and 4(b) respectively. The matrix element of $(A^\dagger)_{\rm eff}$ is then given by the graph in figure 4(a), those in figure 5, which are clearly related to the energy graphs of figure 3, and those of figure 6. To the order considered above, figures 6(a) and 6(b) carry a factor $x \simeq 1$ while 6(c) carries a factor $y \simeq V\Omega/\epsilon$. Evaluation of these graphs gives the result already quoted in equation (21a). A similar analysis for $(A)_{\rm eff}$ reproduces equation (21b).

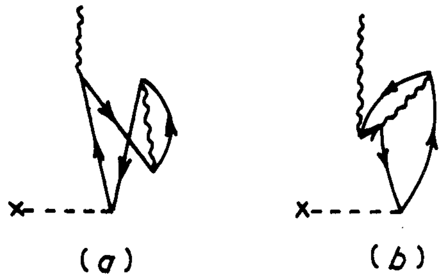

Figure 5. Third order contributions to the specific operator matrix element.

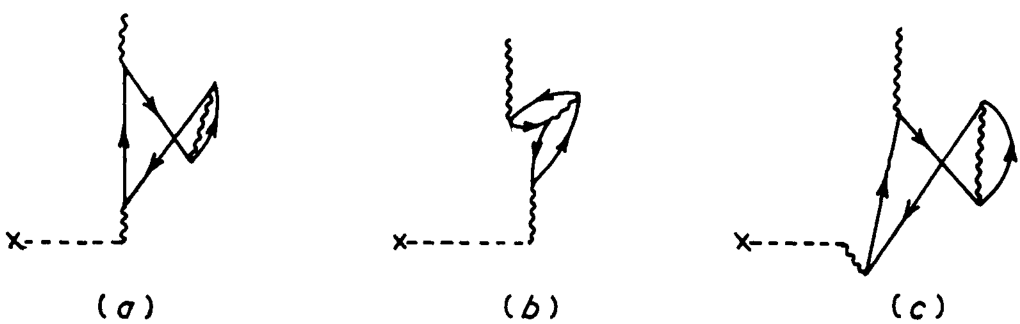

Figure 6. Fourth order contributions to the specific operator matrix element.

It should be emphasized that the inclusion of all these graphs is necessary to obtain agreement with equations (21). A large number of spurious contributions from individual graphs are cancelled by others. It is our conclusion that once a suitable selection of graphs has been made, the effects of redundancy are removed automatically by the particle–vibration interaction which incorporates the true relation between the two modes.

5. Comparison of approximations

We are now in a position to compare the theoretical results of the three boson expansions (HPE, TDE, RPE), and the PVC method. The HPE solves the problem exactly because all the coefficients are known and summed; also the HPE specifies the correct boson basis size (here $2T+1$). In general we can say that if the group underlying a given Hamiltonian is known, then in principle an HPE may be made which will solve the problems exactly, and of course provide truncated boson expansions. If the group is not well known, and therefore the HPE is too difficult to find, then the use of a TDE with the BZ rules will yield a power series in the expansion parameter Ω^{-1}, where Ω is proportional to the number of states available to the active particles. In a general case the relation between the HPE and TDE is that the coefficients of the HPE will contain polynomials of Ω^{-1} corresponding to a partial summation of powers of Ω^{-1} in each term of the series. If these functions are expanded in powers of Ω^{-1}, then the collected terms of an arbitrary power in Ω^{-1} will yield the corresponding term of the TDE result. Therefore, when seeking the HPE of higher groups, the BZ method employing the technically more simple (normal ordered) TDE can provide a guide and a useful check for the lower order terms.

Expanding the RPA equation (27) which defines the phase transition region, gives

$$\frac{\omega_0}{\epsilon} = \left(1-4\frac{V\Omega}{\epsilon}\right)^{1/2} = 1-2\frac{V\Omega}{\epsilon}-2\left(\frac{V\Omega}{\epsilon}\right)^2-4\left(\frac{V\Omega}{\epsilon}\right)^3-\ldots.$$

This may be compared with perturbation theory results made with the TDE, the results with RPE or the graph method, all of which reproduce equation (17) to the particular order treated in Ω^{-1}. We thereby verify the well known theorem that the RPA is an approximation which retains the zeroth order terms in Ω for all orders in the perturbation theory of the collective interaction parameter $V\Omega$, that is, it reproduces all orders of perturbation theory for $V\Omega$ finite and infinite degeneracy. It may have been hoped that the undetermined coefficients of the RPE might constitute extra degrees of freedom through which a more rapid convergence of the series can be obtained. However, the necessity of using the coefficients determined from the RPA as a zeroth order approximation gave results identical to those from the perturbation treatment of the TDE (to the order expanded in Ω^{-1}). That the RPE result is only valid below† (the slightly shifted) phase transition point given by the RPA is simply the statement that the higher order approximation will contain the general features of the zeroth order on which they are based.

The numerical results presented in the next section may be summarized by stating that the diagonalization of a Hamiltonian derived from a truncated TDE (or HPE) gives excellent results below, through and far above the RPA phase transition. Remembering this and pursuing the comparison between the RPE and TDE (or HPE), we are lead to an interesting conclusion; If a consistent boson expansion is made, that is, an exact diagonalization such as RPA or RPE, then the solution will break down as a phase transition is approached. However if a truncated TDE (or HPE) is made to a chosen power in Ω^{-1}, then the diagonalization process in the physical (finite) boson basis in fact includes certain (infinite) series in Ω^{-1}, and these higher powers must be those required to prevent the divergence of the solution near a phase transition. Thus an ambitious consistent expansion in powers of the small parameter Ω^{-1} is doomed because certain higher order (to infinity) terms are necessary for convergence over the entire range of the interaction(s).

As only certain series of powers of Ω^{-1} are included in such a diagonalization, we can expect that qualitative and approximate quantitative arguments (to estimate errors, for example) will hold utilizing the idea that Ω^{-1} is an expansion parameter.

† To go above the phase transition with the RPA, a transformation to quasi-particles could first be made and then the RPA introduced as has been done, eg, by Sorensen (1967).

6. Discussion of numerical results

Exact energy eigenvalues and eigenvectors for the monopole model were obtained by diagonalizing the HPE Hamiltonian (16) in the physical boson space. The matrix for the ground state multiplet is of rank $(\Omega+1)$, and systems with $\Omega = 2, 8$ and 32 are considered. The TD results were obtained by expanding the matrix elements to order Ω^{-1}, Ω^{-2} and Ω^{-3} before diagonalization. Values of the excitation of the first excited state calculated from perturbation theory on TD Hamiltonians (or equivalently from RPE results or the PVC method) are also given for comparison.

The single particle energy ϵ was set equal to unity for all calculations, and $(V\Omega)$ was varied from 2^{7} to 2^{-6}. The RPA phase transition is at $V\Omega = 2^{-2}$.

The exact results are in the first column of each table and the remaining columns if labelled by Ω^{-1}, Ω^{-2} or Ω^{-3} give the TD results to the specified order. The perturbation theory results appear in columns labelled 'first order', 'second order' etc. An entry 'ex' means that the result was exact to within the precision of the calculation.

Tables 1, 2 and 3 show the values of the various calculated quantities as functions of three values of $V\Omega$. A striking result is the accuracy achieved with the $\Omega = 2$ system. For this system it is only in the highly perturbed region that excited state wavefunctions are moderately accurate. In the other regions the approximation is quite good.

Comparison between the Ω^{-1} TDE truncation and first, second and third order perturbation theory can be made by referring to table 4. The perturbation theory results

Table 1. The $\Omega = 2$ system; comparison of calculated quantities as a function of $V\Omega$ and the Ω^{-1}, Ω^{-2}, and Ω^{-3} truncations of a Tamm–Dancoff boson expansion. Notation is discussed in § 3

$V\Omega/\epsilon$		Exact	Ω^{-1}	Ω^{-2}	Ω^{-3}
128	E_0	−384·003	−390·001	−387·950	−386·405
	E_1	−384·003	−387·962	−385·512	−384·594
	E_2	−0·028 297	−0·116 430	−0·092 930	−0·068 754
	$\tilde{n}$	0·997 396	1·00 983	1·00 664	1·00 350
	σ_M	3·99997	4·48213	4·16584	4·07418
	$\sigma_1(g, g)$	0·502 604	0·496 362	0·497 974	0·499 549
	$\sigma_1(g, f)$	0·281 240	0·246 389	0·278 493	0·279 518
	$\sigma_1(g, s)$	0·000 000	0·000 031	0·000 011	0·000 004
$\frac{1}{4}$	E_0	−2·06 096	−2·06 229	−2·06 119	−2·06 103
	E_1	−1·44 300	−1·44 574	−1·44 404	−1·44 341
	E_2	−0·476 329	−0·479 589	−0·478 989	−0·478 100
	$\tilde{n}$	−0·038 553	0·039 455	0·038 763	0·038 643
	σ_M	1·55 124	1·57 310	1·55 625	ex
	$\sigma_1(g, g)$	0·980 015	0·995 837	0·995 931	0·995 946
	$\sigma_1(g, f)$	0·000 726	0·000 758	0·000 733	0·000 729
	$\sigma_1(g, s)$	0·000 708	0·000 795	0·000 720	0·000 707
$\frac{1}{32}$	E_0	−2·00 076	−2·00 077	ex	ex
	E_1	−1·04 797	−1·04 802	−1·04 799	−1·04 798
	E_2	−0·062 454	−0·062 527	−0·062 450	−0·062 489
	$\tilde{n}$	0·000 389	0·000 398	0·000 391	0·000 391
	σ_M	1·04 915	1·05 026	1·04 930	1·04 918
	$\sigma_1(g, g)$	0·999 785	0·999 779	0·999 784	ex
	$\sigma_1(g, f)$	0·000 000	ex	ex	ex
	$\sigma_1(g, s)$	0·000 019	0·000 021	0·000 020	ex

Table 2. The $\Omega = 8$ system; details as for table 1

$V\Omega/\epsilon$		Exact	Ω^{-1}	Ω^{-2}	Ω^{-3}
128	E_0	−1920·01	−1922·00	−1921·07	−1920·60
	E_1	−1920·01	−1922·00	−1921·07	−1920·60
	E_2	−1440·01	−1442·00	−1441·31	−1440·92
	$\tilde{n}$	0·997 917	0·998 953	0·998 727	0·998 501
	σ_M	15·9 999	18·0 796	16·6 265	16·2 443
	$\sigma_1(\mathrm{g}, \mathrm{g})$	0·501 042	0·500 523	0·500 636	0·500 749
	$\sigma_1(\mathrm{g}, \mathrm{f})$	0·490 586	0·491 089	0·490 979	0·490 867
	$\sigma_1(\mathrm{g}, \mathrm{s})$	0·000 000	ex	ex	ex
$\frac{1}{4}$	E_0	−8·11 535	−8·11 549	−8·11 536	ex
	E_1	−7·71 995	−7·72 028	−7·71 999	−7·71 996
	E_2	−7·14 187	−7·14 229	−7·14 197	−7·14 189
	$\tilde{n}$	0·030 446	0·030 494	0·030 452	0·030 447
	σ_M	2·43 057	2·43 673	2·43 118	2·43 067
	$\sigma_1(\mathrm{g}, \mathrm{g})$	0·984 509	0·984 484	0·984 506	0·984 508
	$\sigma_1(\mathrm{g}, \mathrm{f})$	0·001 665	0·001 670	0·001 666	ex
	$\sigma_1(\mathrm{g}, \mathrm{s})$	0·000 267	ex	ex	ex
$\frac{1}{32}$	E_0	−8·00 097	ex	ex	ex
	E_1	−7·06 112	ex	ex	ex
	E_2	−6·11 274	−6·11 275	ex	ex
	$\tilde{n}$	0·000 128	ex	ex	ex
	σ_M	1·06 386	1·06 393	ex	ex
	$\sigma_1(\mathrm{g}, \mathrm{g})$	0·999 934	ex	ex	ex
	$\sigma_1(\mathrm{g}, \mathrm{f})$	0·000 000	ex	ex	ex
	$\sigma_1(\mathrm{g}, \mathrm{s})$	0·000 001	ex	ex	ex

do not exist above the phase transition, are poor near the phase transition ($2^{-2} \lesssim V\Omega \lesssim 2^{-3}$) and quite accurate far below the phase transition ($V\Omega \lesssim 2^{-3}$). Most model calculations are done in the intermediate region (below, but near the phase transition). We would rather emphasize the convergence of the TD expansions over the entire range of the interaction strength and therefore have presented the results on a larger scale. The excellent agreement at the RPA transition represents the intermediate region adequately.

In table 5 we have focused on the accuracy of the Ω^{-1} TDE truncation by presenting relative errors of E_1, $\tilde{n}$, $\sigma_M(\Omega)$ and $\sigma_1(\mathrm{g}, \mathrm{f})$. This truncation is important because it involves the product of no more than four bosons and is thus quite accessible technically. All of the results show that the truncation is capable of describing the monopole model well for arbitrary values of the interaction parameter. Two interesting features are that the relative error showed a fluctuation near the phase transition for the quantity $\tilde{n}$, and that the relative error in $\sigma_M(\Omega)$ approaches a constant ($\simeq 12{\cdot}5\%$, independent Ω) in the high interaction limit. Relative errors of energies and $\sigma_M(\Omega)$ showed a smooth decrease as Ω increased and $V\Omega$ decreased.

In the monopole limit all the relative errors, except for $\sigma_M(\Omega)$, approach constants for a fixed Ω. Assuming, in this large interaction limit, that the relative error $R_q(\Omega)$ of a particular quantity q may be represented to lowest order by the functional form,

$$R_q(\Omega) = r_q \Omega^{-m} \tag{36}$$

Table 3. The $\Omega = 32$ system; details as for table 1

$V\Omega/\epsilon$		Exact	Ω^{-1}	Ω^{-2}	Ω^{-3}
128	E_0	−8064·03	−8064·53	−8064·28	−8064·16
	E_1	−8064·03	−8064·53	−8064·28	−8064·16
	E_2	−7560·03	−7560·53	−7560·30	−7560·18
	$\tilde{n}$	0·998 016	0·998 018	0·998 018	0·998 048
	σ_M	63·9 997	72·0 456	66·2 082	64·7 577
	$\sigma_1(g, g)$	0·500 992	0·500 961	0·500 967	0·500 976
	$\sigma_1(g, f)$	0·497 023	0·497 054	0·497 046	0·497 039
	$\sigma_1(g, s)$	0·000 000	ex	ex	ex
$\frac{1}{4}$	E_0	−32·1 582	ex	ex	ex
	E_1	−31·9 032	−31·9 033	ex	ex
	E_2	−31·5 447	−31·5 448	−31·5 448	−31·5 448
	$\tilde{n}$	0·017 105	0·017 108	ex	ex
	σ_M	3·81 138	3·81 308	3·81 145	ex
	$\sigma_1(g, g)$	0·991 389	0·991 387	ex	ex
	$\sigma_1(g, f)$	0·001 730	ex	ex	ex
	$\sigma_1(g, s)$	0·000 058	ex	ex	ex
$\frac{1}{32}$	E_0	−32·0 010	ex	ex	ex
	E_1	−31·0645	ex	ex	ex
	E_2	−30·1 258	ex	ex	ex
	$\tilde{n}$	0·000 034	ex	ex	ex
	σ_M	1·06 773	1·06 774	ex	ex
	$\sigma_1(g, g)$	0·999 983	ex	ex	ex
	$\sigma_1(g, f)$	0·000 000	ex	ex	ex
	$\sigma_1(g, s)$	0·000 000	ex	ex	ex

Table 4. Energy differences between the ground and first excited states calculated from the Ω^{-1} TDE truncation, and first, second and third order perturbation theory

		$E_1 - E_0$				
Ω	$V\Omega/\epsilon$	Exact	Ω^{-1}	First order	Second order	Third order
2	$\frac{1}{4}$	0·617	0·616	0·625	0·625	0·601
	$\frac{1}{8}$	0·808	0·807	0·812	0·812	0·806
	$\frac{1}{16}$	0·904	ex	0·906	0·906	ex
	$\frac{1}{32}$	0·952	ex	0·953	0·953	ex
	$\frac{1}{64}$	0·976	ex	ex	ex	ex
8	$\frac{1}{4}$	0·395	ex	0·531	0·437	0·403
	$\frac{1}{8}$	0·737	ex	0·765	0·742	ex
	$\frac{1}{16}$	0·876	ex	0·882	ex	ex
	$\frac{1}{32}$	0·939	ex	0·941	ex	ex
	$\frac{1}{64}$	0·970	ex	ex	ex	ex
32	$\frac{1}{4}$	0·254	ex	0·507	0·390	0·336
	$\frac{1}{8}$	0·715	ex	0·753	0·724	0·717
	$\frac{1}{16}$	0·868	ex	0·876	0·869	ex
	$\frac{1}{32}$	0·936	ex	0·938	ex	ex
	$\frac{1}{64}$	0·968	ex	0·969	ex	ex

Table 5. The relative errors in the Ω^{-1} truncation as a function of Ω and $V\Omega/\epsilon$ for the quantities E_1, $\tilde{n}$, σ_M and $\sigma_1(\mathrm{g, f})$

$V\Omega/\epsilon$	$\{(E_1)_{\Omega^{-1}}-(E_1)_{\mathrm{ex}}\}/(E_1)_{\mathrm{ex}}$ $\Omega=2$	$\Omega=8$	$\Omega=32$	$\{(\tilde{n})_{\Omega^{-1}}-(\tilde{n})_{\mathrm{ex}}\}/(\tilde{n})_{\mathrm{ex}}$ $\Omega=2$	$\Omega=8$	$\Omega=32$
128	1.03×10^{-2}	1.03×10^{-3}	6.1×10^{-5}	1.24×10^{-2}	1.03×10^{-2}	6.1×10^{-5}
64	1.03×10^{-2}	1.03×10^{-3}	6.1×10^{-5}	1.24×10^{-2}	1.03×10^{-2}	6.1×10^{-5}
32	1.03×10^{-2}	1.02×10^{-3}	6.1×10^{-5}	1.25×10^{-2}	1.03×10^{-2}	6.1×10^{-5}
16	1.02×10^{-2}	1.00×10^{-3}	6.0×10^{-5}	1.25×10^{-2}	1.02×10^{-2}	6.1×10^{-5}
8	1.02×10^{-2}	9.73×10^{-4}	5.8×10^{-5}	1.26×10^{-2}	1.00×10^{-2}	6.0×10^{-5}
4	1.01×10^{-2}	7.84×10^{-4}	5.4×10^{-5}	1.29×10^{-2}	0.97×10^{-2}	5.8×10^{-5}
2	9.53×10^{-3}	5.67×10^{-4}	4.7×10^{-5}	1.37×10^{-2}	0.92×10^{-2}	5.5×10^{-5}
1	7.81×10^{-3}	2.53×10^{-4}	3.4×10^{-5}	1.78×10^{-2}	0.87×10^{-2}	5.2×10^{-5}
$\frac{1}{2}$	4.67×10^{-3}	4.2×10^{-5}	1.6×10^{-5}	2.53×10^{-2}	1.00×10^{-2}	5.5×10^{-5}
$\frac{1}{4}$	1.89×10^{-3}	7×10^{-6}	1×10^{-6}	2.33×10^{-2}	1.57×10^{-2}	5.3×10^{-5}
$\frac{1}{8}$	5.97×10^{-4}	1×10^{-6}	$<10^{-6}$	2.15×10^{-2}	1.11×10^{-2}	6.7×10^{-5}
$\frac{1}{16}$	1.65×10^{-4}	$<10^{-6}$	$<10^{-6}$	2.10×10^{-2}	1.05×10^{-2}	6.2×10^{-5}
$\frac{1}{32}$	4.3×10^{-5}	$<10^{-6}$	$<10^{-6}$	2.08×10^{-2}	1.04×10^{-2}	6.2×10^{-5}
$\frac{1}{64}$	1.1×10^{-5}	$<10^{-6}$	$<10^{-6}$	2.08×10^{-2}	1.04×10^{-2}	6.2×10^{-5}
$V\Omega/\epsilon$	$\{(\sigma_M)_{\Omega^{-1}}-(\sigma_M)_{\mathrm{ex}}\}/(\sigma_M)_{\mathrm{ex}}$ $\Omega=2$	$\Omega=8$	$\Omega=32$	$\{(\sigma_1(\mathrm{g,f}))_{\Omega^{-1}}-(\sigma_1(\mathrm{g,f}))_{\mathrm{ex}}\}/(\sigma_1(\mathrm{g,f}))_{\mathrm{ex}}$ $\Omega=2$	$\Omega=8$	$\Omega=32$
128	1.20×10^{-1}	1.29×10^{-1}	1.25×10^{-1}	1.72×10^{-1}	1.02×10^{-3}	6.2×10^{-5}
64	1.19×10^{-1}	1.29×10^{-1}	1.24×10^{-1}	1.70×10^{-1}	1.02×10^{-3}	6.2×10^{-5}
32	1.18×10^{-1}	1.27×10^{-1}	1.24×10^{-1}	1.66×10^{-1}	1.01×10^{-3}	6.0×10^{-5}
16	1.16×10^{-1}	1.24×10^{-1}	1.20×10^{-1}	1.61×10^{-1}	1.01×10^{-3}	6.1×10^{-5}
8	1.12×10^{-1}	1.18×10^{-1}	1.15×10^{-1}	1.43×10^{-1}	9.96×10^{-2}	6.0×10^{-5}
4	1.03×10^{-1}	1.08×10^{-1}	1.04×10^{-1}	1.11×10^{-1}	9.72×10^{-2}	5.7×10^{-5}
2	8.87×10^{-2}	8.89×10^{-2}	8.60×10^{-2}	3.57×10^{-3}	9.30×10^{-2}	5.7×10^{-5}
1	6.55×10^{-2}	5.92×10^{-2}	5.69×10^{-2}	1.44×10^{-2}	8.90×10^{-2}	5.1×10^{-5}
$\frac{1}{2}$	3.66×10^{-2}	2.31×10^{-2}	2.13×10^{-2}	4.15×10^{-2}	1.20×10^{-3}	5.7×10^{-5}
$\frac{1}{4}$	1.40×10^{-2}	2.53×10^{-3}	4.46×10^{-3}	4.40×10^{-2}	2.97×10^{-3}	2.71×10^{-4}
$\frac{1}{8}$	5.38×10^{-3}	4.11×10^{-4}	3.2×10^{-4}	4.30×10^{-2}	2.20×10^{-3}	1.33×10^{-4}
$\frac{1}{16}$	2.30×10^{-3}	1.61×10^{-4}	1.0×10^{-4}	4.20×10^{-2}	2.14×10^{-3}	1.26×10^{-4}
$\frac{1}{32}$	1.06×10^{-3}	6×10^{-5}	5×10^{-5}	4.21×10^{-2}	2.09×10^{-3}	1.24×10^{-4}
$\frac{1}{64}$	5×10^{-4}	3×10^{-5}	2×10^{-5}	4.21×10^{-2}	2.08×10^{-3}	1.23×10^{-4}

then we may find from table 5 that $m \simeq 2$, verifying that the error in the Ω^{-1} truncation is of the order Ω^{-2}.

7. Conclusions

Two of the boson expansions examined (HPE and TDE) produced excellent values for the energies and several physical quantities over the entire range of the interaction parameter V including the phase transition. In contrast, a third boson expansion (RPE) and the particle–vibration coupling method yielded results equivalent to perturbation theory and were therefore valid only below the phase transition.

The particle–vibration coupling results, obtained by evaluation of all contributing graphs and fixing the parameters of the method from the RPA, were in agreement with perturbation theory for both the energy and specific matrix element. From this it may be inferred that the problem of overcompleteness of the basis has been eliminated and furthermore, the exclusion principle restrictions are satisfied provided that the phonon is defined according to the RPA (equation (31)) and not the quasi-boson approximation.

After finishing this work we noted that Klein (1971) has discovered that the TDE holds for arbitrary V in the Lipkin model. Since he does not present details we have included a coverage of the TDE for comparison with the other approximations.

References

Beliaev S T and Zelivinsky V G 1962 *Nucl. Phys.* **39** 582–604

Bès D R and Dussel G G 1969 *Nucl. Phys.* A **135** 1–24

Evans J A and Kraus N C 1972 *Phys. Lett.* **37B** 455–9

Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098–113

Janssen D, Donau F and Frauendorf S 1971 *Nucl. Phys.* A **172** 145–65

Klein A 1971 *Theories of Collective Motion*, Lectures given at the Mont Tremblant International Summer School Chap II-E

Lane A M 1964 *Nuclear Theory* (New York: Benjamin) chap 3, 8

Li S Y, Dreizler R M and Klein A 1971 *Phys. Rev.* C **4** 1571–91

Lipkin H J *et al* 1965 *Nucl. Phys.* **62** 188–98

Marshalek E R 1971 *Nucl. Phys.* A **161** 401–9

Marumori T, Yamamura M and Tokunaga A 1964 *Prog. theor. Phys.* **31** 1009–25

Mottelson B R 1968 *J. Phys. Soc. Japan, Suppl.* **24** 87–98

Pang S C, Klein A and Dreizler R M 1968 *Ann. Phys.* **49** 477–95

Sanderson E A 1965 *Phys. Lett.* **19** 141–3

Sorensen B 1967 *Nucl. Phys.* A **97** 1–32

—— 1968 *Nucl. Phys.* A **119** 65–78

—— 1969 *Nucl. Phys.* A **142** 392–410